

Confidence Sets in Time-Series Filtering

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Abstract

The problem of filtering of finite-alphabet stationary ergodic time series is considered. A method for constructing a confidence set for the (unknown) signal is proposed, such that the resulting set has the following properties: First, it includes the unknown signal with probability γ , where γ is a parameter supplied to the filter. Second, the size of the confidence sets grows exponentially with the rate that is asymptotically equal to the conditional entropy of the signal given the data. Moreover, it is shown that this rate is optimal. We also show that the described construction of the confidence set can be applied for the case where the signal is corrupted by an erasure channel with unknown statistics.

1 Introduction

The problem of estimating a discrete signal X_1, \dots, X_t from a noisy version Z_1, \dots, Z_t has attracted attention of many researchers due to its great importance for statistics, computer science, image processing, astronomy, biology, cryptography, information theory and many other fields. The main attention is usually focused on developing methods of estimation (denoising, or filtering) of the unknown signal, with the performance measured under a given fidelity criterion; see [8, 9] and references therein. Such an approach is close in spirit to the problem of point estimation in statistics.

An alternative approach, often considered in mathematical statistics, is that of constructing confidence sets. That is, one tries to use the data to construct a set that includes the unknown parameter (in our case, the signal) with a prescribed probability, while trying to keep the size of the set as small as possible (some classical examples of the use of this method in statistics can be found in, e.g., [4]). Such a set is usually constructed as the set of most likely values of the parameter.

The reason why such an approach is of interest is as follows. In the presence of noise, the exact recovery of the signal is typically impossible, and thus, in such cases, any of its estimates is necessarily imperfect. The choice of a particular estimate of the signal out of many likely alternatives is largely arbitrary. Moreover, the optimal choice may depend on the specific application involved. The confidence-set approach effectively abstracts from the problem of choosing the “best” estimate, proposing, instead, a set of estimates. The performance

of a method is then characterized by the size of the confidence set (depending on the confidence level). This is the approach and the problems considered in this work. We consider a model in which the underlying noiseless signal and the resulting corrupted (noisy) signal (and thus the channel) are assumed to be stationary ergodic processes with finite alphabets. We mainly concentrate on the case where the probability distributions of the noiseless signal and the noisy channel are known. (Obviously, in such a case the distribution of the corrupted signal is known, too.) Besides, the case of a erasure channel with unknown distribution is briefly mentioned, because in this case a conditional distribution of noiseless signal is known even though the distribution of the noise is unknown. The results that we obtain establish the optimal rate of growth (with respect to time, or to the length of the signal) of the size of the confidence set, as well as a method for constructing such a set. The optimal rate turns out to be equal to the entropy of the signal given its noisy version.

Let us consider an example that illustrates our approach and exposes the notation. Let the signal be binary (with the alphabet $\{0, 1\}$), and suppose that it is transmitted through a memoryless binary erasure channel (e.g. [1]). The binary erasure channel with erasure probability π is defined as a channel with binary input, ternary output (with the alphabet $\{0, 1, *\}$), and the probability of erasure π . The channel replaces each input symbol 0 or 1 with the (output) symbol $*$ with probability π (erasure), and places the input signal in the output otherwise (that is, with probability $1 - \pi$).

Suppose that the noiseless sequence is generated by an i.i.d. source P and $P\{X_i = 0\} = 0.9$, and let the erasure probability be any $\pi \in (0, 1)$, i.e. the erasure probability is unknown. Suppose that the corrupted by noise sequence is as follows:

$$Z_1 \dots Z_4 = 0 * 1 * .$$

Then we have the following probability distribution for the lossless signal:

$$P(\{X_1 \dots X_4 = 0010\}) = 0.81,$$

$$P(\{X_1 \dots X_4 = 0110\}) = 0.09,$$

$$P(\{X_1 \dots X_4 = 0011\}) = 0.09,$$

$$P(\{X_1 \dots X_4 = 0111\}) = 0.01.$$

If we take the confidence level $\gamma = 0.99$, the confidence set will contain three following sequences: $\{0010, 0110, 0011\}$.

The goal of this paper is to describe a construction of confidence sets and to give an estimate of their size, for the case when the signal and noise are stationary ergodic processes with finite alphabets. It is shown that for any $\gamma \in (0, 1)$ the size of the confidence set grows exponentially with the rate $h(X|Z)$, where $h(X|Z)$ is the limit (conditional) Shannon entropy. This result is valid for the case when the probability distributions of noiseless signal and noise are known as well as for the case when the probability distribution of the signal is known and the noise is described by a stationary erasure channel with memory

whose probability distribution is unknown. Moreover, we prove that the rate $h(X|Z)$ is minimal, which means that the suggested method of constructing confidence sets is asymptotically optimal.

It is worth noting that the information theory is deeply connected with statistics of time series and signal processing; see, for example, [1, 2, 5, 7, 10, 11, 13, 14, 15] and [8, 9, 6], correspondingly. In this paper a new connection of this kind is established: it is shown that the Shannon entropy determines the rate of growth of the size of the confidence set for the signal, given its version corrupted by stationary noise.

2 The confidence sets and their properties

We consider the case where the signal $X = X_1, X_2, \dots$ and its noisy version $Z = Z_1, Z_2, \dots$ are described by stationary ergodic processes with finite alphabets \mathbf{X} and \mathbf{Z} respectively. (There may be arbitrary long-range dependencies between the variables.) It is assumed that probability distributions of both processes are known, and, hence, the statistical structure of the noise corrupting the signal $X = X_1, X_2, \dots$ is known, too. Introduce the short-hand notation $X_{1..t}$ for X_1, \dots, X_t , and analogously for Z .

Informally, for any $\gamma \in (0, 1)$ and any sequence Z_1, \dots, Z_t we define the confidence set $\Psi_\gamma^t(Z_1, Z_2, \dots, Z_t)$ as follows: the set contains sequences x_1, x_2, \dots, x_t whose probabilities $P(x_{1..t}|Z_{1..t})$ are maximal and sum to γ . This definition is not precise, since it is possible that the sum can not be made equal to γ exactly. That is why the formal definition of the confidence set will use randomization.

For this purpose, we order all sequences $X_{1..t}$ according their conditional probabilities, in the decreasing order. That is, enumerate all sequences $x_{1..t} \in \mathbf{X}^n$ in such a way that $(a_{1..t}) \in \mathbf{X}^t$ has a smaller index than $(b_{1..t}) \in \mathbf{X}^t$ if either $P(a_{1..t}|Z_{1..t}) > P(b_{1..t}|Z_{1..t})$, or $P(a_{1..t}|Z_{1..t}) = P(b_{1..t}|Z_{1..t})$ and $(a_{1..t})$ is lexicographically less than $(b_{1..t})$. Let j be the integer for which $\sum_{i=1}^{j-1} P(x_{1..t}^i|Z_{1..t}) \leq \gamma$ and $\sum_{i=1}^j P(x_{1..t}^i|Z_{1..t}) > \gamma$. If $\sum_{i=1}^{j-1} P(x_{1..t}^i|Z_{1..t}) = \gamma$, then define $\Psi_\gamma^t(Z_{1..t})$ as the set $\{x_{1..t}^1, \dots, x_{1..t}^{j-1}\}$. Otherwise, $\Psi_\gamma^t(Z_{1..t})$ also contains $j - 1$ first elements, and additionally the element $x_{1..t}^j$ with probability $(\gamma - \sum_{i=1}^{j-1} P(x_{1..t}^i|Z_{1..t}))/P(x_{1..t}^j|Z_{1..t})$. (Note that this procedure is commonly used in mathematical statistics for making the confidence level exactly γ .) When talking about the sizes of the confidence sets we refer to their expected (with respect to the randomization) size.

Next, we estimate the size of the described confidence set.

Theorem 1. *Let an (unknown) signal $X = X_1 X_2, \dots$ and its noisy version $Z = Z_1 Z_2, \dots$ be stationary ergodic processes with finite alphabets. Then, for every $\gamma \in (0, 1)$, all $t \in \mathbb{N}$ and almost every Z_1, \dots, Z_t the confidence set $\Psi_\gamma^t(Z_1, \dots, Z_t)$ contains the unknown (X_1, \dots, X_t) with probability γ :*

$$P\{X_{1..t} \in \Psi_\gamma^t(Z_{1..t})\} = \gamma, \quad (1)$$

while, with probability 1, the size of the set $\Psi_\gamma^t(Z_1, \dots, Z_t)$ grows exponentially with the exponent rate that is equal to the conditional entropy:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbf{E} |\Psi_\gamma^t(Z_1, \dots, Z_t)| = h(X|Z) \text{ a.s.}, \quad (2)$$

where the expectation is with respect to the randomization used in constructing the confidence sets.

Proof. The proof of (1) immediately follows from the construction of the set $\Psi_\gamma^t(Z_1 Z_2 \dots Z_t)$.

The proof of (2) will be based on the Shannon-McMillan-Breiman theorem [1, 3], which for the conditional entropy implies the following:

Lemma 1 (Shannon-McMillan-Breiman). $\forall \varepsilon > 0, \forall \delta > 0$, for almost all Z_1, Z_2, \dots there exists n' such that if $n > n'$ then

$$P \left\{ \left| -\frac{1}{n} \log P(X_{1..n}|Z_{1..n}) - h(X|Z) \right| < \varepsilon \right\} \geq 1 - \delta. \quad (3)$$

Take any $\varepsilon > 0$ and any $\delta > 0$ such that

$$1 - \delta \geq \gamma. \quad (4)$$

According to the lemma, for almost all Z_1, Z_2, \dots there exists n' such that (3) is valid if $n > n'$. Take any such n and rewrite (3) as follows:

$$P \left\{ 2^{-n(h(X|Z)+\varepsilon)} \leq P(X_{1..n}|Z_{1..n}) \leq 2^{-n(h(X|Z)-\varepsilon)} \right\} \geq 1 - \delta. \quad (5)$$

Thus, the probability of all strings x_1, \dots, x_n for which we have $P(x_{1..n}|Z_{1..n}) \geq 2^{-n(h(X|Z)+\varepsilon)}$ is at least $(1 - \delta)$. Taking into account (4), we have

$$|\Psi_\gamma^t(Z_{1..n})| \leq \gamma / 2^{-n(h(X|Z)+\varepsilon)},$$

so that

$$\frac{1}{n} \log |\Psi_\gamma^t(Z_{1..n})| \leq h(X|Z) + \varepsilon + O(1/n) \quad (6)$$

for $n > n'$. Having taken into account that (6) holds for every small $\varepsilon > 0$ we obtain (2). \square

3 Optimality of the confidence set

Theorem 2. Let an (unknown) signal $X = X_1 X_2, \dots$ and its noisy version $Z = Z_1 Z_2, \dots$ be stationary ergodic processes with finite alphabets \mathbf{X} and \mathbf{Z} . Let $\Phi_\gamma^t(Z_{1..t})$, be confidence sets, such that for some $\gamma \in (0, 1)$ we have $P(X_{1..t} \in \Phi_\gamma^t(Z_{1..t})) \geq \gamma$ for all $t \in \mathbb{N}$ and almost all $Z_{1..t} \in \mathbf{Z}^t$. Then, with probability 1,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log |\Phi_\gamma^t(Z_1, \dots, Z_t)| \geq h(X|Z). \quad (7)$$

Proof. The proof will use the Shannon-McMillan-Breiman theorem (5). As before, we take any $\varepsilon > 0$ and fix $\delta := \gamma/2$. Then from some n on we have (5). Let Υ be a confidence set for this n and a certain γ . Define

$$\Phi = \left\{ x_{1..n} : 2^{-n(h(X|Z)+\varepsilon)} \leq P(x_{1..n}|Z_{1..n}) \leq 2^{-n(h(X|Z)-\varepsilon)} \right\}. \quad (8)$$

By definition, $\sum_{x_{1..n} \in \Upsilon} P(x_{1..n}|Z_{1..n}) \geq \gamma$. From this and (5) we obtain

$$\sum_{x_{1..n} \in \Upsilon \cap \Phi} P(x_{1..n}|Z_{1..n}) \geq \gamma - \delta.$$

From this and (8) we get

$$|\Upsilon| \geq |\Upsilon \cap \Phi| \geq (\gamma - \delta) 2^{n(h(X|Z)-\varepsilon)}.$$

Hence,

$$\liminf_{t \rightarrow \infty} \frac{1}{n} \log |\Upsilon| \geq h(X|Z) - \varepsilon.$$

Since this inequality is true for any confidence set Υ and any $\varepsilon > 0$, we obtain (7). \square

4 Erasure channel with unknown statistics

In this section we consider the case when the channel statistics is unknown, but the channel has a specific form: it is an erasure channel for which probabilities to be erased are equal for all symbols. We show that the described above confidence set is asymptotically optimal in this case, too. The point is that in this the conditional probabilities $P(X_{1..n}/Z_{1..n})$ are known, that is why the construction of the previous section is directly applicable.

The formal description of the considered model is as follows. We still assume that there is a known stationary ergodic source generating the signal X_1, X_2, \dots . The erasure channel is defined in two following steps: first, there is a stationary ergodic process Θ generating letters from the alphabet $\{\Lambda, *\}$ and, second, the noisy channel is determined by the following “summation” of the (uncorrupted) sequence X_1, X_2, \dots and the noise sequence $\Theta_1, \Theta_2, \dots$:

$$Z_i = \begin{cases} X_i & \text{if } \Theta_i = \Lambda \\ * & \text{if } \Theta_i = *. \end{cases}$$

Theorem 3. *Let an (unknown) signal $X = X_1 X_2, \dots$ and Z_1, Z_2, \dots be a stationary ergodic signal and its version corrupted by an unknown stationary erasure channel. Then, for every $\gamma \in (0, 1)$, all $t \in \mathbb{N}$ and almost every Z_1, \dots, Z_t the (above described) confidence set $\Psi_\gamma^t(Z_1, \dots, Z_t)$ contains the unknown (X_1, \dots, X_t) with probability γ :*

$$P\{X_{1..t} \in \Psi_\gamma^t(Z_{1..t})\} = \gamma, \quad (9)$$

while, with probability 1, the size of the set $\Psi_\gamma^t(Z_1, \dots, Z_t)$ grows exponentially with the exponent rate that is equal to the conditional entropy:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbf{E} |\Psi_\gamma^t(Z_1, \dots, Z_t)| = h(X|Z) \text{ a.s.}, \quad (10)$$

where the expectation is with respect to the randomization used in constructing the confidence sets.

Proof. It is enough to notice that, although the erasure channel is not known, the probabilities $P(X_{1..n}|Z_{1..n})$ are known. Therefore, the proof of this theorem is identical to that of Theorem 1. \square

5 Discussion

To the best of our knowledge, the problem of constructing a confidence set for the unknown signal was not considered before, which is why there are many quite natural and obvious extensions and generalizations of the present work. First, it is interesting to consider this problem for certain specific classes of distributions of the signal and noise, such as i.i.d. and Markov sources. For these classes of sources it should be possible to obtain rates of convergence in those statements that in this work are only asymptotic, for example in (2).

Second, a natural question is to find a construction of the confidence set for the cases where the signal is multi-dimensional. This is particularly important for applications, many of which are concerned with denoising such objects as photographs or video fragments. Another interesting generalization is the case where the alphabets are (subsets of), for example, the Euclidean space. This generalization can be also interesting from the practical point of view. Finally, the case where statistics of the noise and/or signal are unknown is obviously of great theoretical and practical interest.

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References

- [1] Thomas M. Cover and Joy A. Thomas. *Elements of information theory*. Wiley-Interscience, New York, NY, USA, 2006.

- [2] I. Csiszar and P.C. Shields. Notes on information theory and statistics. In *Foundations and Trends in Communications and Information Theory*, 2004.
- [3] R. G. Gallager. *Information Theory and Reliable Communication*. John Wiley & Sons, New York, NY, USA, 1968.
- [4] M.G. Kendall and A. Stuart. *The advanced theory of statistics; Vol.2: Inference and relationship*. London, 1961.
- [5] G. Morvai, S. Yakowitz, and L. Györfi. Nonparametric inference for ergodic, stationary time series. *Ann. Statist.*, 24(1):370–379, 1996.
- [6] E. Ordentlich, G. Seroussi, S. Verdu, and K. Viswanathan. Universal algorithms for channel decoding of uncompressed sources. *Information Theory, IEEE Transactions on*, 54(5):2243–2262, May 2008.
- [7] J. Rissanen. Universal coding, information, prediction, and estimation. *Information Theory, IEEE Transactions on*, 30(4):629–636, July 1984.
- [8] J. Rissanen. MDL denoising. *IEEE Transactions on Information Theory*, 46(7):2537–2543, November 2000.
- [9] T. Roos, P. Myllymaki, and J. J. Rissanen. MDL denoising revisited. *IEEE Transactions on Signal Processing*, 57(9):3347–3360, 2009.
- [10] B. Ryabko. Compression-based methods for nonparametric prediction and estimation of some characteristics of time series. *IEEE Transactions on Information Theory*, 55:4309–4315, 2009.
- [11] B. Ryabko and J. Astola. Universal codes as a basis for time series testing. *Statistical Methodology*, 3:375–397, 2006.
- [12] B. Ryabko, D. Ryabko, Confidence Sets in Time - Series Filtering. *Proceedings of 2011 IEEE International Symposium on Information Theory (ISIT'11)*, July, 31 - August, 5, 2011, in Saint-Petersburg, Russia
- [13] D. Ryabko. Testing composite hypotheses about discrete-valued stationary processes. In *Proc. IEEE Information Theory Workshop (ITW'10)*, pages 291–295, Cairo, Egypt, 2010. IEEE.
- [14] D. Ryabko. Testing composite hypotheses about discrete ergodic processes. *Test*, page (to appear), 2011.
- [15] D. Ryabko and B. Ryabko. Nonparametric statistical inference for ergodic processes. *IEEE Transactions on Information Theory*, 56(3):1430–1435, 2010.